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Alternative Models of the Real Number Line in Physics

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Mathematicians have shown that alternative models of the real number line exist, which have all properties of the usual real numbers (a complete ordered field). Since our conception of metric distances in space-time can be put in one-to-one relation with our conception of the real number line, if alternative models exist, it becomes a matter for experiment which model nature has chosen for physical space-time. We consider one such model in which ordinary real numbers are replaced by random variables and show that the model satisfies the 11 axioms of a complete ordered field. This model then is used to model metric distances in physical space-time and to explore the physical implications for the propagation of light. A nice feature of the model is that it provides a new conceptual framework for building a constant with dimensions of length into space-time. The Mössbauer effect restricts this fundamental length to be less than 10^{-24} cm.

1. INTRODUCTION

Toward the end of the last century, Riemann and other mathematicians developed self-consistent models of geometry which were non-Euclidean. This caused quite a stir in mathematical circles at the time, but only Einstein fully realized that if more than one mathematical formalism for geometry exists then it becomes a matter for experiment to decide which formalism best describes the geometry of physical space-time. Einstein's general relativity is now supported by numerous experiments and physical space-time in fact seems to be non-Euclidean.

More recently, since 1963, another revolution has taken place in mathematics which is probably more fundamental in importance than the earlier geometrical revolution. Mathematicians have shown that there are alternative models for the real number line which are quite different in content from the standard model, and yet satisfy all the axioms for a

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complete ordered field, which defines what we mean by a real number line (Steen, 1971). There are two general types of alternative models. The first was developed by Robinson (1961, 1966) to incorporate actual infinitesimals into the real number line. This has led to a formalism called nonstandard analysis, which has recently been applied to physics by Kelemen and Robinson (1972a, b). This model has the same physical content as the standard model and it has been shown that it is conservative in the sense that one can never obtain a result which could not have been obtained with standard analysis. The advantage of nonstandard analysis is that some calculations are much easier to do this way than with the usual analysis. It should be noted that the Robinson model is an ordered field but not a complete ordered field. Since completion involves second-order sentences while the other ten axioms for an ordered field involve first-order sentences and since most of calculus involves only first-order sentences, for most purposes completion is unnecessary.

The other type of alternative model was developed by Cohen (1966) and by Scott (1967). This model arose from the desire to construct a model of the real number line which contains a set which violates Cantor's continuum hypothesis (the statement that every infinite set of real numbers is either countable or of cardinality c). Models of this type are now used extensively in mathematics and represent a complete ordered field. The objects of both types of alternative model are real-valued functions f defined on some set S. The alternative real number system then becomes R^S, which is the set of all functions from S to R where R is the usual real number line. One axiom defining a field is that for any real number f except zero, there is another real number f^{-1} that satisfies $ff^{-1} = u$ where u is the unit (existence of a multiplicative inverse). In the present context, if we have some functions f that do not have inverses, we still obey the axiom if we redefine truth so that "f equals zero" will be true. More precisely, we substitute a probabilistic notion for truth or validity for the usual deterministic one. For example, "f equals g" where f and g are members of our alternative real number system becomes the measure m of the set S for which $f(s) = g(s)$ or $|f = g| = m(\{s \in S | f(s) = g(s)\})$. "$f$ equals g" is *valid* if this probability is one. Even though $f(s)$ might not equal $g(s)$ for a large number of points $s \in S$, "f equals g" would be valid so long as the measure of this large number of points were zero. R^S/m with this new concept of validity is always an ordered field. For some choices of S and m it will also be a complete ordered field. For the Robinson model mentioned above, S is the set of positive integers and m the cofinite measure ($m = 0$ for finite sets, $m = 1$ for cofinite sets, and $m = 0$ or 1 for intermediate sets in some consistent way). For the Scott model $S = I^T$ which is the set of all functions from T (a set whose cardinal number is larger than c) to I (the unit interval) and m is an extension of the Lebesque measure. This latter model is clearly vastly larger and richer than the usual model of the real numbers.

In our work below we choose a particular alternative model of the general Cohen–Scott type and use it to model space-time metric distances. Physical space-time metric distances may be modeled by an alternative model rather than the standard model in which case our present conception of space and time may be fundamentally wrong. It becomes a matter of experiment which model of the real number line nature is chosen, especially if different models lead to different physical predictions. We show below that the particular R^S/m model we investigate makes predictions for the propagation of light different from the predictions of the standard model of the real number line. (For comparison, the nonstandard analysis of Robinson makes the same predictions as the usual real number line.) This arises from the probabilistic interpretation of validity mentioned above. When such a probabilistic model is substituted for the usual real number line, physical space-time metric distances themselves become probabilistic random variables, with physical consequences. This is highly reminiscent of the transition from classical to quantum physics but at a more basic level.

The particular model we look at below uses a probability measure which involves a fundamental constant length σ. We will find that σ profoundly modifies the short distance behavior of space-time. Thus one way in which this work can be viewed is as a revolutionary way of building a fundamental length into space-time, which is quite different from early work on discrete space and space-time (Heisenberg, 1938; Snyder, 1947; Hellund and Tanaka, 1954; Schild, 1949; Hill, 1955; Lanczos, 1964, 1966; Peters, 1974) or the more recent work on lattice gauge theories (Wilson, 1974; Kogut, 1979; Drouffe and Itzykson, 1978).

We discuss our probabilistic model of the real number line in Section 2 below and prove that it satisfies the axioms for a complete ordered field. In Section 3 we consider this model of the real number line as modeling physical space-time metric distances. We apply our new model to physics in Section 4 by investigating the propagation of light. Finally in Section 5 we briefly summarize our findings.

2. PROBABILISTIC MODEL OF THE REAL NUMBER LINE

The particular model of the real number line we wish to consider is the simple model in which *each element* $x \in R$ of the usual real number line (which still has a role to play) has a random variable $\chi(s, x)$ associated with it where $s \in S$ and the probability space S will also be taken to be R. To complete the description we also need a measure on S (actually on a sigma field of subsets of S) which will be taken to be

$$m(\{s \in S | \beta \leqslant \chi(s,x) \leqslant \alpha\}) = \int_\beta^\alpha \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma} e^{-(s-x)^2/2\sigma^2} ds \qquad (1)$$

The random variable $\chi(s, x)$ is thus a real-valued function defined on the sample space S with x denoting the mean of the random variable. [More technically, $\chi(s, x)$ defined on S is a random variable if for every Borel set B in the real line R, the set $\{s; \chi(s,x) \in B\}$ is in F, a σ field of subsets of S.] If $x = 3$, for example, (1) gives the probability distribution for the random variable associated with $x = 3$ of the usual real number line. This random variable has a mean of 3 and a variance of σ. This measure gives the size of intervals $[\beta, \alpha]$ on S and is normalized to 1 when the interval is taken to include the entire space S. *The random variables $\chi(s, x)$ will be the elements of our new real number line.* σ is a fundamental constant length which measures the size of the spread of the probability distribution. As $\sigma \to 0$ the integrand in (1) becomes a delta distribution and the random variable $\chi(s, x)$ essentially reduces back to x. In this limit, our new model of the real number line reduces back to the usual model. Note that the constant σ is taken to be the *same* for all the random variables associated with all the different points of the original real number line. Thus $\chi(s, y)$ would be the random variable associated with $y \in R$, where y is a different point from x. It would have the same measure m as in (1) but with y substituted for x. Thus our new real number line is a separate copy of S at *every* point in the usual real number line and is far larger than R. Note that since the random variables $\chi(s, x)$ associated with different points x have the same σ, we are not considering the set of *all* functions from $S \to R$ as the R^S/m models do. Our model will nonetheless be shown to satisfy the 11 axioms for a complete ordered field below.

The above model was chosen for simplicity. A normal distribution was chosen because of the statistical fact that whatever the distribution of a set of physical "measurements," the distributions of many functions of them tend to normality when the number of "measurements" tends to infinity. Using a normal probability distribution and the same σ at every point of R is suggested by the physical picture of a space-time which is foamlike for very short distances with all points treated equivalently. Such a foamlike picture arises naturally from quantum effects in general relativity for distances of the order of the Planck length $(\hbar G/c^3)^{1/2} = 1.6 \times 10^{-33}$ cm (Misner et al., 1973). Our foam is of a different character than theirs, however, arising from the smearing of metric distances rather than from topological changes. Presumably our σ is comparable to the Planck length in size, although this is by no means necessary. (Our σ has no relation to the infinitesimals of Robertson's work.) The present work can then be viewed as providing a new conceptual framework for considering such a foamlike structure. Although simplicity leads one rather naturally to our particular model, many other R^S/m-type models exist, any one of which might be a better description of physical space and time. What is needed is a general principle which picks out a particular real number line just like the Hilbert action picks out a particular geometry in general relativity. Lacking such a principle, we can only explore plausible models.

Before we show that our model satisfies all the axioms of a complete ordered field, we must discuss the ordering of the elements of our real number line $\chi(s', x)$ and $\chi(s, y)$. This ordering must needs be probabilistic in nature. Using (1) we have

$$P(\chi(s',x)>\chi(s,y))$$
$$= \int_{-\infty}^{\infty}\left[\int_{s}^{\infty}\frac{1}{(2\pi)^{1/2}}\frac{1}{\sigma}e^{-(y-s)^2/2\sigma^2}\frac{1}{(2\pi)^{1/2}}\frac{1}{\sigma}e^{-(x-s')^2/2\sigma^2}\,ds'\right]ds \tag{2}$$

for the probability that $\chi(s', x) > \chi(s, y)$. If we define $\alpha \equiv (y-x)/\sqrt{2}\,\sigma$ we can put (2) into the form

$$P(\chi(s',x)>\chi(s,y)) = \frac{1}{2} - \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-2\zeta}\operatorname{erf}(\alpha+\zeta)\,d\zeta \tag{3}$$

where erf is the error function. Integrating this for various values of α shows that

$$P(\chi(s',x)>\chi(s,y)) \text{ is } \begin{array}{ll} >1/2 & \text{if } x>y \\ =1/2 & \text{if } x=y \\ <1/2 & \text{if } x<y \end{array} \tag{4}$$

We also have

$$P(\chi(s',x)>\chi(s,y)) + P(\chi(s',x)<\chi(s,y)) = 1 \tag{5}$$

as we would expect. σ determines how far the probability distributions are spread out. If $y-x=\sqrt{2}\,\sigma$, for example, we find $P(\chi(s',x)>\chi(s,y)) \simeq 0.125$ so that $\chi(s', x)$ is greater than $\chi(s, y)$ 1/8 of the time even though y

is greater than x by $\sqrt{2}\,\sigma$. If $y - x = 3\sqrt{2}\,\sigma$, $P(\chi(s', x) > \chi(s, y))$ would already be quite small.

Given any two random variables (elements of our new real number line), $\chi(s', x)$ and $\chi(s, y)$, we can work out $P(\chi(s', x) > \chi(s, y))$ and use (4) to establish trichotomy with a probabilistic interpretation of validity. Thus if $P(\chi(s', x) > \chi(s, y))$ is $1/2$ we say the two "real numbers" $\chi(s', x)$ and $\chi(s, y)$ are equal. If this probability is $> 1/2$ we say $\chi(s', x)$ is greater than $\chi(s, y)$ and if this probability is $< 1/2$ we say $\chi(s', x)$ is less than $\chi(s, y)$, thus establishing trichotromy.

Having defined equality and established trichotomy, we are now in a position to show that our model satisfies the 11 requirements for a complete ordered field. We define addition and multiplication of our alternative real numbers by

$$\chi(s, x) + \chi(s, y) \equiv \chi(s, x + y) \tag{6}$$

and by

$$\chi(s, x) \cdot \chi(s, y) \equiv \chi(s, xy) \tag{7}$$

where x and y are usual real numbers. Note that (6) could have been written equally well as

$$\chi(s, x) + \chi(s', y) \equiv \chi(s'', x + y) \tag{6a}$$

I use the single letter s in (6) and (7) and also everywhere below for simplicity of notation and to emphasize that all the random variables have the *same* probability distribution associated with them with the *same* constant σ. Thus (6a) means that the sum of alternative real numbers $\chi(s, x)$ and $\chi(s', y)$ is taken to be the alternative real number (random variable) whose mean is $x + y$ and whose probability distribution has the same measure and the same variance σ as the probability distributions of $\chi(s, x)$ and $\chi(s', y)$ themselves. Equality among our alternative real numbers here and below is always understood in the probabilistic sense as explained above following (5), i.e., $\chi(s, x) = \chi(s, y)$ if and only if $P(\chi(s, x) > \chi(s, y)) = 1/2$. $\chi(s, x + y)$ and $\chi(s, xy)$ both clearly represent alternative real numbers, also, satisfying the first requirement. The associativity, commutativity, and distributivity (requirements 2, 3, and 4) of our alternative real numbers follow very easily from (6) and (7). The fifth requirement is that there exist two special numbers z(zero) and u(unit) that satisfy $\theta + z = \theta$ and $\theta u = \theta$ for any generalized real number θ. In the present case, let $\chi(s, 0)$ be the zero and $\chi(s, 1)$ be the unit. From (6) and (7) we clearly have

$$\chi(s,x)+\chi(s,0)=\chi(s,x+0)\equiv\chi(s,x) \tag{8}$$

and

$$\chi(s,x)\cdot\chi(s,1)=\chi(s,x) \tag{9}$$

as required. The additive inverse of $\chi(s,x)$ can be taken to be $\chi(s,-x)$ where we have

$$\chi(s,x)+\chi(s,-x)=\chi(s,0) \qquad \text{(our zero)} \tag{10}$$

satisfying property 6. Similarly, the multiplicative inverse of $\chi(s,x)$ can be taken to be $\chi(s,x^{-1})$. This satisfies

$$\chi(s,x)\cdot\chi(s,x^{-1})=\chi(s,1) \qquad \text{(our unit)} \tag{11}$$

satisfying property 7. Property 8 is trichotomy: we have already shown above that given two of our real numbers $\chi(s',x)$ and $\chi(s,y)$, that we can consistently say whether $\chi(s',x)<\chi(s,y)$, $\chi(s',x)=\chi(s,y)$, or $\chi(s',x)>\chi(s,y)$ in a probabilistic sense. Transitivity (property 9) followed directly from (4) and from the fact that $\chi(s',x)>\chi(s,y)$ in our probabilistic sense if and only if $x>y$ for the ordinary real numbers x and y. Since ordinary real numbers obey transitivity, so will $\chi(s',x)$ and $\chi(s,y)$. Additive and multiplicative isotony (property 10) can be argued similarly: Assume $\chi(s,x)<\chi(s,y)$ in our probabilistic sense. Then $x<y$ from (4). Since ordinary real numbers obey additive isotony, we have $x+w<y+w$ where w is any other ordinary real number. Again using (4), this implies $\chi(s,x+w)<\chi(s,y+w)$ or from (6) we have

$$\chi(s,x)+\chi(s,w)<\chi(s,y)+\chi(s,w) \tag{12}$$

where $\chi(s,w)$ is any of our alternative real numbers, as we wanted to show. Multiplicative isotony follows similarly. For the eleventh and last property, completion, we want to show that if a set of our alternative real numbers has an upper bound, then it has a least upper bound. This is shown in the Appendix and relies on the fact that ordinary real numbers obey completion, and (4) establishes a correspondence between ordinary real numbers and our alternative real numbers.

We conclude that our simple probabilistic model satisfies all eleven requirements for a complete ordered field and thus is a viable alternative model of the real number line. It is different from the usual model in that (1) it is based on a probabilistic interpretation of equality and inequality and (2) a constant σ is built in.

3. PHYSICAL SPACE TIME

We now want to use our alternative real numbers to model space-time. The only place that the usual real number line plays a role in our description of space-time is in the metric properties of the space-time. We will be interested in a Riemannian manifold (Choquet-Bruhat et al., 1977) which is a smooth manifold χ (assumed finite dimensional here) together with a continuous 2-covariant tensor field **g**, the metric tensor. **g** is symmetric, and for each $x \in \chi$, the bilinear form $\mathbf{g}_x$ is nondegenerate so that $\mathbf{g}_x(\mathbf{V}, \mathbf{W}) = 0$ for all $\mathbf{V} \in T_x$ (the tangent vector space) iff $\mathbf{W} = 0$. Thus the metric **g** maps two elements of the tangent vector space onto the real number line, yielding the dot product of the vectors. We will model these real numbers using our alternative model of the real number line. We can state this another way which more directly exhibits the space-time properties. If we introduce local coordinates, the metric tensor components $g_{\alpha\beta}$ can be used to define the length of a piece of curve $\times^\alpha(P)$ with $P_0 \leqslant P \leqslant P_1$ to be

$$S = \int_{P_0}^{P_1} \left[g_{\alpha\beta} \frac{d\times^\alpha}{dP} \frac{d\times^\beta}{dP} \right]^{1/2} dP \tag{13}$$

The length S is a real number and we will model these real numbers using our alternative model.

Representing the metric lengths S in (13) by our alternate model of the real number line means that the actual length of a given path whose nominal or average length is a given number S becomes a random variable ξ where the probability that ξ will lie between the values α and β is given by

$$\int_\beta^\alpha \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma} e^{-(\xi-S)^2/2\sigma^2} \, d\xi \tag{14}$$

ξ can take any value but only values of ξ within several σ of S are probable. We will explore in the next section what limits experiments put on the fundamental length σ.

Let us see with the above modifications whether we still have a true metric space. Very generally, a set M is called a metric space (Kuratowski, 1966) if for each pair of elements $x, y \in M$, the distance $d(x, y)$ is defined, and satisfies

$$d(x, y) \geqslant 0 \text{ with } d(x, y) = 0 \text{ iff } x = y \tag{15}$$

$$d(x, y) = d(y, x) \tag{16}$$

$$d(x, z) \leqslant d(x, y) + d(y, z) \tag{17}$$

If these equalities and inequalities are all understood in the *probabilistic sense* of our earlier discussion in (4) then we still have a metric space.

We will be interested in the following in the propagation of light since it makes a nice example of how our alternative model of the real number line can affect physics. Light travels a "null" geodesic so that we will assume that the nominal or average length of its path in space-time given by (13) is zero. The actual length of the null geodesic is then the random variable ξ with a probability

$$\int_{\beta}^{\alpha} \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma} e^{-\xi^2/2\sigma^2} d\xi \tag{18}$$

of being between α and β. The mean value of ξ is

$$\int_{-\infty}^{\infty} \xi \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma} e^{-\xi^2/2\sigma^2} = 0 \tag{19}$$

as we would expect. Thus the light cone is smeared out in a probabilistic sense by several σ and we have lost strict microscopic causality on length scales of the order of σ. On the average, causality is still preserved, but it is not preserved in detail on these length scales. This smearing of light cones due to our alternative model of the real number line has experimental consequences which we explore in the next section. We will try to put experimental limits on the size of the fundamental length σ.

4. EXPERIMENTAL CONSEQUENCES

Let us look at the propagation of light more closely. If we specialize to flat space for simplicity, we have

$$dS^2 = -\mathbf{d}x^2 + c^2 dt^2 \tag{20}$$

where dS^2 is the square of the length of the infinitesimal line element. We can rearrange this to yield

$$\beta^2 = 1 - \left(\frac{dS}{c\,dt}\right)^2 \tag{21}$$

where β is the actual coordinate velocity of light divided by c. If we let $L = c\Delta t$ be the spatial length of the light path and let ΔS be given by the random fluctuation ξ in the length of the space-time geodesic, we have approximately

$$\beta^2 = 1 - \frac{\xi 2}{L^2} \tag{22}$$

We get a significant change in β only for spatial paths whose lengths are comparable to σ, i.e., very small. β is a measure of how far the light is off the mass shell. One can envision testing (22) directly using laser pulses of very short duration. The random fluctuation σ will produce a random spreading of the arrival time of photons in a pulse containing many photons. Thus the time width of the packet can be related to ξ and can thus be used to put a limit on the fundamental length σ. For observations over a spatial length L, the difference in arrival times of two photons, one of which is traveling with β given by (22) and the other traveling with $\beta = 1$ is

$$\Delta t = \frac{L}{\beta c} - \frac{1}{c} \approx \frac{\xi^2}{2Lc} \tag{23}$$

assuming $\xi^2/L^2 \ll 1$. Averaging over many photons using (18) gives

$$(\Delta t)_{\text{average}} \approx \int_{-\infty}^{\infty} \frac{\xi^2}{2Lc} \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma} e^{-\xi^2/2\sigma^2}\, d\xi \tag{24}$$

or

$$(\Delta t)_{\text{average}} \approx \frac{\sigma^2}{2Lc} \tag{25}$$

for the time width of the packet due to these effects. If we have a laser pulse with duration $\Delta t \approx 10^{-12}$ sec and observe it over $L = 10$ cm we have $\sigma \lesssim 0.8$ cm, which is not restrictive enough to be interesting. We need a better way to restrict σ.

β in (22) is really a measure of how far the photon is off the "mass shell" or "energy shell." Thus the photon's energy or frequency varies because of (22). We can estimate the size of this effect by writing the total photon energy as

$$E = \frac{M_{\text{eff}} c^2}{(1 - \beta^2)^{1/2}} \tag{26}$$

where M_{eff} is the effective mass of the photon. Using (22) then gives

$$M_{\text{eff}} c^2 \approx \frac{\xi}{L} E \tag{27}$$

If we let $M_{\text{eff}} c^2$ represent how far the photon is off the mass shell and write

$M_{\text{eff}}c^2 \approx h\,\Delta\nu$ with $E \approx h\nu$, we have finally

$$\frac{\Delta\nu}{\nu} \approx \frac{\xi}{L} \tag{28}$$

This should be a crude measure of how far the photon is off the mass shell due to the effects of our alternative model of the real number line. (The energy fluctuations we are discussing violate ultramicroscopic causality and also energy conservation for length scales on the order of σ. This should not trouble the reader too much since σ may be quite small, on the order of the Planck length $\approx 10^{-33}$ cm. At this distance scale it is known that quantum effects make the geometry a topological foam with causality and energy conservation meaningless. For length scales larger than this but smaller than elementary particle compton wavelengths, it is a matter of future experiment to see if detailed ultramicroscopic energy conservation holds.) We can use (28) to put a limit on σ using the Mössbauer effect. Notice that $\Delta\nu/\nu = 0$ if we average over many photons as we might expect. Now Mössbauer photons from Fe^{57} have an intrinsic frequency width of $\Delta\nu/\nu \approx 10^{-12}$. If observations can be made over a spatial length of 1 cm between absorber and emitter, (28) would imply that ξ must be smaller than about 10^{-12} cm for single photons. Thus the fundamental length $\sigma < 10^{-12}$ cm also. If we focus on the absorption process or emission process itself and argue that the relevant length scale for L in (28) is the diameter of the Fe^{57} nucleus, then we find $\sigma < 10^{-24}$ cm which is a much more interesting limit.

5. SUMMARY AND CONCLUSIONS

We have written down a particular alternative model of the real number line and have shown that it satisfies the 11 axioms for a complete ordered field. This model was chosen with an eye toward physical applications. Our primary purpose was to show that alternative models of the real number line may model physical space-time better than the conventional model and to demonstrate that such probabilistic models may be very important to physics. Other models exist besides the one we used, and some of these will undoubtedly fit physical space-time better than our model. Ultimately some physical principle must be used to select the correct model just as the Hilbert action chooses the correct physical geometry in general relativity.

In using our alternative model of the real number line to model physical space-time, we found that we built a constant σ, with dimensions of length, into space-time in a very reasonable way. Various investigators have attempted to do this for a long time. An alternative real number line in fact provides a very rich new conceptual framework for this and other problems.

In the final section above, we studied the effects of our alternative model of the real number line on the propagation of light. We found that both the velocity of propagation and frequency might be expected to fluctuate. The Mössbauer effect was then used to put a limit of $\sigma < 10^{-24}$ cm on the allowed size of our fundamental length.

APPENDIX

We want to show that if a set of our alternative real numbers has an upper bound, then it has a least upper bound (completion property). Consider a set E of our alternative real numbers $\{\chi(s, y), \chi(s, z), \cdots\}$ with an upper bound $\chi(s, x_U)$ so that $\chi(s, y) < \chi(s, x_U)$ (in our probabilistic sense) for all $\chi(s, y) \in E$. Now (4) allows us to associate with each member $\chi(s, y)$ of E a corresponding ordinary real number y since $\chi(s, y) < \chi(s, z)$ (in our probabilistic sense) if $y < z$. These corresponding ordinary real numbers $\{y, z, \cdots\}$ can be thought of as belonging to a set of ordinary real numbers F. x_U will clearly be an upper bound to set F. Now since F is comprised of ordinary real numbers which satisfy completion, there must be a least upper bound x_L to the set F, such that $y < x_L$ for all $y \in F$ and $x_L \leqslant x_U$ for all upper bounds x_U. But $y < x_L \Rightarrow \chi(s, y) < \chi(s, x_L)$ for all $\chi(s, y) \in E$ and since $x_L \leqslant x_U \Rightarrow \chi(s, x_L) \leqslant \chi(s, x_U)$, where $\chi(s, x_U)$ is any upper bound for E, $\chi(s, x_L)$ is clearly a least upper bound for our set of alternative real numbers E as we wanted to show.

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